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PERFORMANCE OF MEDIAN FILTERS WITH RANDOM INPUTS

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ABSTRACT

Several empirical studies have shown that median filtering can be an effective nonlinear signal processing technique which is sometimes superior to linear signal processing. In this paper some statistical properties of median filters are analyzed. In particular, results are presented on the noise suppression performance of median filters.

I. INTRODUCTION

Historically, there have been numerous developments concerned with the linear filtering of random sequences. However, in many cases if we restrict our processing of a random sequence to be linear, then the resultant performance is not adequate. Thus in some situations we are led to consider nonlinear schemes for processing data. In several applications of discrete time signal processing, a nonlinear scheme called median filtering has achieved some very interesting results. The implementation of a median filter requires a very simple nonlinear operation. Consider a fixed nonnegative integer N. At a given time instant, the output of the median filter is the empirical median of the samples lying within a window centered at the given time instant and spanning 2N+1 adjacent samples.

Tukey [1] is generally credited with introducing the concept of median filters, and he did much of the pioneering work in this area [2-5]. This idea immediately triggered the use of more sophisticated schemes using recursively or iteratively the sample medians and combining the median operation with some linear processing of the input. Some of these modified median filters are briefly described by Mallows [6] and are also considered by Velleman [7].

Although the concept of median filters was first introduced in the statistics literature, it soon found applications in the area of signal processing. Loosely speaking, linear filtering tends to smooth out abrupt changes in a sequence of data, but a median filter is capable of preserving sharp changes in data. A review of median filters as well as several algorithms for their real-time implementation is presented in [8] and [9], and some applications of median filters are presented in [10-14]. A deterministic analysis of the process of median filtering is presented in [15]. The second moment properties of the output of a median filter with an independent identically distributed input sequence are considered in [16]; and in [17] second moment properties of a median filtered Markov chain are considered. In [18] a comparative study of median filters and Hanning filters in their removal of Gaussian and impulsive noise from a constant signal is presented.

In this paper we will consider both the median filter as described above and also a slight modification of it. These will be defined in the next section. Then we will analyze the effect of a median filter on an input which consists of the sum of a known constant signal and independent identically distributed noise. Specifically, we will study the probability that the difference between the output of the filter and the constant signal is close to zero. This probability can be used to determine the performance of the filter on a per-sample basis. Some examples are presented to illustrate the results.

II. PRELIMINARIES

Consider a discrete time random process $\{X_n, n=0,\pm 1,\pm 2,\ldots\}$ which will denote the input to the median filter. The random process $\{Y_n, n=0, \pm 1,\pm 2,\ldots\}$ will denote the output. Let W be a positive integer. If W is an odd integer, say W = 2N+1, then

$$Y_j = \text{med}\{X_{j-N}, X_{j-N+1},\ldots,X_{j+N}\},$$

where med$\{\cdot\}$ denotes the empirical median. If W is an even integer, say W = 2N, then we will define the output Y_j of the (modified) median filter to be the arithmetic mean of the N-th and (N+1)-st smallest samples in the set $\{X_{j-N+1}, X_{j-N+2},\ldots, X_{j+N}\}$. That is, when W is even, Y_j is the average of the two central order statistics of the samples in the window spanning the W adjacent samples starting at the (j-W/2+1)-st sample. In general, the integer W will be called the window size of the median filter. The sequence $\{Y_j\}$ obtained by positioning the window and selecting the appropriate output value is the output of the median filter. It is clear that, in general, the median filter is a nonlinear scheme. The only exceptions, in which the filter is linear, are when W=1 and

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W=2. A median filter with an odd window size will be called an odd span median filter (OMF) and one with an even window size will be called an even span median filter (EMF). Both OMF's and EMF's involve a ranking procedure, but EMF's also involve an additional averaging.

A study of the dependency introduced by OMF's when the input is a sequence of independent identically distributed random variables or a homogeneous Markov chain is presented in [16] and [17], respectively. Illustrative examples in these cases suggest that OMF's display a typical "low pass" characteristic with large window sizes having narrower "pass bands" than OMF's with small window spans.

In the sequel we will model the input as a known constant signal with additive independent identically distributed noise. That is, let $X_n = a + Z_n$, where $\{Z_n\}$ is a sequence of independent identically distributed random variables. Denote the common univariate distribution function of the noise by F. We will assume that the noise possesses a density function, denoted by $f(x) = dF(x)/dx$. In order to evaluate the noise removal properties of median filters, we will be interested in determining $P(|Y_n - a| \le \varepsilon)$. Without loss of generality, we will simplify notation by assuming that a=0. That is, if an input Z_n is transformed by a median filter to the output $\hat{Z}_n$ and if a is a constant, then the input Z_n+a will be transformed by the median filter to $\hat{Z}_n+a$. Thus if the input is a constant signal plus noise, then the output minus the constant signal is simply what the output would be if the input had consisted of noise alone.

III. ODD SPAN MEDIAN FILTERS

Let $W = 2N+1$. Then

$$Y_j = \text{med}\{X_{j-N}, X_{j-N+1}, \ldots, X_{j+N}\}.$$

Since the input sequence $\{X_n\}$ is independent and identically distributed, we see that the output $\{Y_j\}$ is identically distributed (but not independent). We recall that the density function of Y_j, say f_Y, is given by [19, p.9]

$$f_Y(y) = \frac{(2N+1)!}{(N!)^2} f(y)[F(y)]^N[1-F(y)]^N. \quad (1)$$

We are interested in

$$P(|Y_j| \le \varepsilon) \quad (2)$$

for $\varepsilon > 0$. Using the binomial expansion and a change of variable, we can see that substitution of (1) into (2) yields

$$P(|Y_j| \le \varepsilon) = \frac{(2N+1)!}{(N!)^2} \sum_{i=0}^{N} \binom{N}{i}(-1)^{N-i} \cdot \left[\frac{[F(\varepsilon)]^{2N+1-i} - [F(-\varepsilon)]^{2N+1-i}}{2N+1-i}\right]. \quad (3)$$

Thus we see that for a given W = 2N+1 and a given $\varepsilon > 0$, $P(|Y_j| \le \varepsilon)$ depends only upon the values of $F(\varepsilon)$ and $F(-\varepsilon)$. For example, let ε be a positive number less than one. Let $q_1(\varepsilon) = P(|Y_j| \le \varepsilon)$ when X_n is uniformly distributed on $[-1,1]$. For $\tilde{\varepsilon}$ positive, let $q_2(\tilde{\varepsilon}) = P(|Y_j| \le \tilde{\varepsilon})$ when X_n has a Laplace distribution with zero mean and unit variance, and let $q_3(\tilde{\varepsilon}) = P(|Y_j| \le \tilde{\varepsilon})$ when X_n has a Gaussian distribution with zero mean and unit variance. Then a straightforward calculation shows that

$$q_1(\varepsilon) = q_2\left[-\frac{\sqrt{2}}{2}\ \ell n(1-\varepsilon)\right]$$

$$= q_3\left[\Phi^{-1}\left(\frac{1+\varepsilon}{2}\right)\right],$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du.$

The expression in (3) furnishes a convenient method for calculating the probability of error associated with a median filter. For example, if $\{X_n\}$ is a sequence of independent zero mean unit variance Gaussian random variables, then for a median filter with window size W=3, $P(|Y_j| \le 0.1) \approx 0.119$, and for a median filter with window size W=5, $P(|Y_j| \le 0.1) \approx 0.149$, while $P(|X_n| \le 0.1) \approx 0.096$.

A plot of $P(|Y_j| \le \varepsilon)$ versus $F(\varepsilon)$ is given in Fig. 1 for symmetric noise densities and different window sizes.

Consider for the moment a one sided noise density, i.e. $f(x)=0$ for $x < 0$. Then (3) reduces to

$$P(|Y_j| \le \varepsilon) = \frac{(2N+1)!}{(N!)^2} \sum_{i=0}^{N} \binom{N}{i} \frac{(-1)^{N-i}}{2N+1-i}[F(\varepsilon)]^{2N+1-i}.$$

If we realize that in this case, $P(|Y_j| \le \varepsilon)$ is simply the probability that at least N+1 samples within the window are no greater than ε, we see that the above expression can be re-written to give

$$P(|Y_j| \le \varepsilon) = \sum_{i=N+1}^{2N+1} \binom{2N+1}{i}[F(\varepsilon)]^i[1-F(\varepsilon)]^{2N+1-i}$$

when $f(x)=0$ for $x < 0$. A straightforward calculation yields that for a median filter with window size $W=3$ if $F(0)=0$ and $F(\varepsilon) = 0.1$, then $P(|Y_j| \leq \varepsilon) = 0.028$; while if $W=5$, we have $P(|Y_j| \leq \varepsilon) = 0.00856$. However, in these cases the median filter did not help, since $P(|X_n| \leq \varepsilon) = 0.1$.

IV. EVEN SPAN MEDIAN FILTERS

In the previous section it was necessary for the value of one of the input samples from within the window to lie within the interval $[-\varepsilon,\varepsilon]$ in order for the event $(|Y_j| \leq \varepsilon)$ to occur. In the case of an EMF the event $(|Y_j| \leq \varepsilon)$ could occur even if none of the input samples lie within the interval $[-\varepsilon,\varepsilon]$. As described previously, for an EMF the event $(|Y_j| \leq \varepsilon)$ is equivalent to the situation that, after ranking the $W=2N$ samples, the arithmetic mean of the two central order statistics has a magnitude no greater than ε. To calculate the probability of this event, we will use the bivariate density function of the two central order statistics, and we will then do a transformation to obtain the density function of the output of the EMF.

Let $W=2N$ be the window size, and let S_1 and S_2 be the two central order statistics. That is, of the $2N$ samples in the window, S_1 is the N-th and S_2 is the $(N+1)$-st in increasing order. Then the bivariate density of S_1 and S_2 is given by [19, p. 10]

$$f_{S_1,S_2}(s_1,s_2) = \frac{(2N)!}{[(N-1)!]^2} f(s_1)f(s_2) \cdot$$
$$\cdot [F(s_1)]^{N-1}[1-F(s_2)]^{N-1}$$

for $s_1 \leq s_2$, and it is zero for $s_1 > s_2$. Define the random variable S by

$$S = \frac{1}{2}(S_1 + S_2).$$

Then the bivariate density of S_1 and S is straightforwardly found to be

$$f_{S_1,S}(s_1,s) = \frac{2N^2(2N)!}{(N!)^2} f(s_1)f(2s-s_1) \cdot$$
$$\cdot [F(s_1)]^{N-1}[1-F(2s-s_1)]^{N-1}$$

when $s \geq s_1$, and it is zero when $s < s_1$. Thus we see that the density function of the output of the EMF is given by

$$f_S(s) = \frac{2N^2(2N)!}{(N!)^2} \int_{-\infty}^{s} f(x)f(2s-x) \cdot$$
$$\cdot [F(x)]^{N-1}[1-F(2s-x)]^{N-1}dx.$$

Upon integrating $f_S(s)$ over $[-\varepsilon,\varepsilon]$, we obtain $P(|Y_j| \leq \varepsilon)$.

V. EXAMPLE

Consider zero mean unit variance noise with the Laplace distribution function

$$F(x) = \begin{cases} \frac{1}{2} e^{\sqrt{2}x}, & x \leq 0 \\ \frac{1}{2}[2-e^{-\sqrt{2}x}], & x > 0 \end{cases}.$$

The performance of a median filter for this input distribution was calculated according to the developments of Sections III and IV. For $\varepsilon > 0$ and W odd ($W=2N+1$) we have

$$P(|Y_j| \leq \varepsilon) = \frac{(2N+1)!}{(N!)^2} \sum_{i=0}^{N} \binom{N}{i} (-1)^{N-i}$$
$$\cdot \left[\frac{(2-e^{-\sqrt{2}\varepsilon})^{2N+1-i} - e^{-\sqrt{2}\varepsilon(2N+1-i)}}{(2N+1-i)2^{2N+1-i}}\right].$$

For $\varepsilon > 0$ and W even ($W=2N$) we have

$$P(|Y_j| \leq \varepsilon) = 1-2F_S(-\varepsilon),$$

where

$$F_S(-\varepsilon) = \int_{-\infty}^{-\varepsilon} f_S(s)ds$$

and

$$f_S(s) = \frac{N^2(2N)!}{(N!)^2\, 2^{2N-2}}\left[\int_{-\infty}^{2s} \exp[2N\sqrt{2}(y-s)]dy\right.$$
$$\left.+\int_{2s}^{s} \exp[\sqrt{2}(y[N-1]-2s)]\cdot[2-\exp(-\sqrt{2}(y+2s))]^{N-1}dy\right]$$

for $s \leq 0$.

Fig. 2 indicates the performance characteristics of median filters with window sizes $W=1,2,3,4,5$ when the input is independent witn a zero mean unit variance Laplace distribution.

VI. COMMENTS

Recall that a median of a random variable X is any number m such that

$$P(X \geq m) \geq \frac{1}{2} \leq P(X \leq m).$$

Thus a random variable either has a unique median, or else it has medians for all points of a closed interval [20, p. 256]. Assume for the moment that the distribution function F, the common distribution function of the input sequence of independent random variables, admits a unique median m. Then

it follows in a straightforward fashion from the Glivenko-Cantelli theorem [21, p. 261] that as the window size W approaches infinity, the output of the median filter converges to m with probability one. Thus if $X_n \geq 0$ with probability one and if $\varepsilon < m$, then it is easily seen that $P(|Y_j| \leq \varepsilon)$ should converge to zero as W approaches infinity. This observation is consistent with the phenomenon exhibited in the example at the end of Section III. Conversely, for a zero mean Gaussian or Laplace input, we would expect that for any positive ε, $P(|Y_j| \leq \varepsilon)$ should converge to one as W approaches infinity; and this observation is consistent with the examples we have exhibited.

In conclusion, we have considered the noise suppression performance of median filters when the input consists of a constant signal plus independent identically distributed noise. The results presented in this paper enable one to determine the performance of a median filter on a per-sample basis. Several examples were presented to illustrate the results; and it was seen that in some cases median filters did provide noise suppression, while in other cases (e.g. one sided noises) they did not help. Further investigations of the noise suppression performance of median filters for nonconstant signals in dependent noise would be useful.

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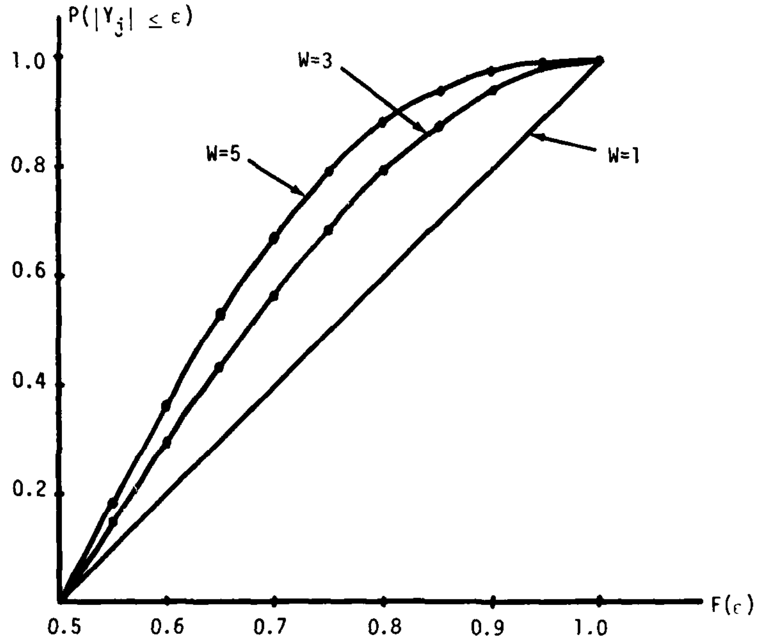

Figure 1: $P(|Y_j| \le \varepsilon)$ versus $F(\varepsilon)$ for a symmetric input distribution.

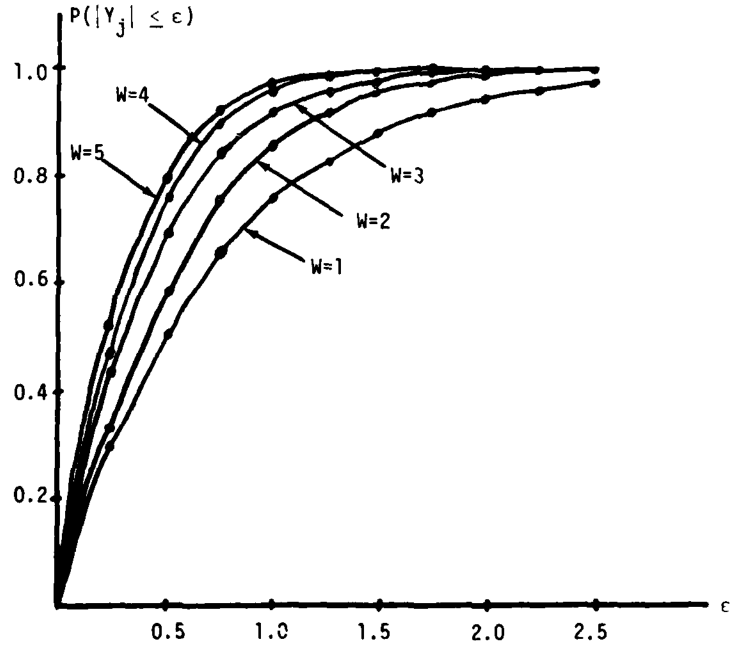

Figure 2: Performance for unit variance Laplace noise.

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